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DIFFERENTIAL EQUATIONS WITH *MATHEMATICA*®

THIRD EDITION

MATHEMATICA 5

INCLUDES CD-ROM

Martha L. Abell & James P. Braselton

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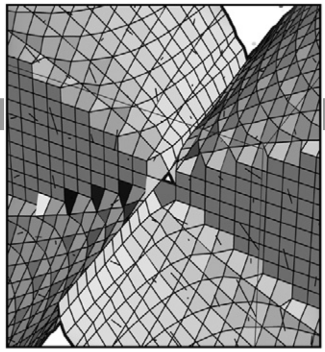

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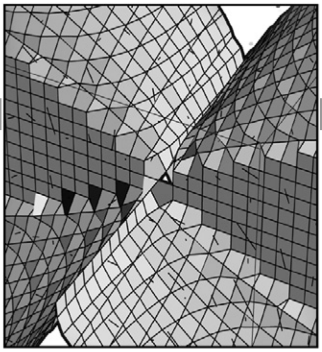
Preface



Mathematica's diversity makes it particularly well suited to performing many calculations encountered when solving many ordinary and partial differential equations. In some cases, Mathematica's built-in functions can immediately solve a differential equation by providing an explicit, implicit, or numerical solution; in other cases, Mathematica can be used to perform the calculations encountered when solving a differential equation. Because one goal of elementary differential equations courses is to introduce students to basic methods and algorithms and have the student gain proficiency in them, nearly every topic covered in *Differential Equations with Mathematica*, Third Edition, includes typical examples solved by traditional methods and examples solved using Mathematica. *Differential Equations with Mathematica* introduces basic commands and includes typical examples of applications of them. A study of differential equations relies on concepts from calculus and linear algebra so the text also includes discussions of relevant commands useful in those areas. In many cases, seeing a solution graphically is most meaningful so *Differential Equations with Mathematica* relies heavily on Mathematica's outstanding graphics capabilities.

Differential Equations with Mathematica is an appropriate reference for all users of Mathematica who encounter differential equations in their profession, in particular, for beginning users like students, instructors, engineers, business people, and other professionals using Mathematica to solve and visualize solutions to differential equations. *Differential Equations with Mathematica* is a valuable supplement for students and instructors at engineering schools that use Mathematica.

Taking advantage of Version 5 of Mathematica, *Differential Equations with Mathematica*, Third Edition, introduces the fundamental concepts of Mathematica to solve (analytically, numerically, and/or graphically) differential equations of interest to students, instructors, and scientists. Other features to help make *Differential Equations with Mathematica*, Third Edition, as easy to use and as useful as possible include the following.

1. **Version 5 Compatibility.** All examples illustrated in *Differential Equations with Mathematica*, Third Edition, were completed using Version 5 of Mathematica. Although most computations can continue to be carried out with earlier versions of Mathematica, like Versions 2, 3, and 4, we have taken advantage of the new features in Version 5 as much as possible.
2. **Applications.** New applications, many of which are documented by references, from a variety of fields, especially biology, physics, and engineering, are included throughout the text.
3. **Detailed Table of Contents.** The table of contents includes all chapter, section, and subsection headings. Along with the comprehensive index, we hope that users will be able to locate information quickly and easily.
4. **Additional Examples.** We have considerably expanded the topics in Chapters 1 through 6. The results should be more useful to instructors, students, business people, engineers, and other professionals using Mathematica on a variety of platforms. In addition, several sections have been added to help make locating information easier for the user.
5. **Comprehensive Index.** In the index, mathematical examples and applications are listed by topic, or name, as well as commands along with frequently used options: particular mathematical examples as well as examples illustrating how to use frequently used commands are easy to locate. In addition, commands in the index are cross-referenced with frequently used options. Functions available in the various packages are cross-referenced both by package and alphabetically.
6. **Included CD.** All Mathematica code that appears in *Differential Equations with Mathematica*, Third Edition, is included on the CD packaged with the text.
7. **Getting Started.** The Appendix provides a brief introduction to Mathematica, including discussion about entering and evaluating commands, loading packages, and taking advantage of Mathematica's extensive help facilities. Appropriate references to *The Mathematica Book* are included as well.

We began *Differential Equations with Mathematica* in 1990 and the first edition was published in 1991. Back then, we were on top of the world using Macintosh IIcx's with 8 megs of RAM and 40 meg hard drives. We tried to choose examples that we thought would be relevant to typical users — typically in the context of differential equations encountered in the undergraduate curriculum. Those examples could also be carried out by Mathematica in a timely manner on a computer as powerful as a Macintosh IIcx.

Now, we are on top of the world with Power Macintosh G4's with 768 megs of RAM and 50 gig hard drives, which will almost certainly be obsolete by the time you are reading this. The examples presented in *Differential Equations with Mathematica* continue to be the ones that we think are most similar to the problems encountered by beginning users and are presented in the context of someone familiar with mathematics typically encountered by undergraduates. However, for this third edition of *Differential Equations with Mathematica* we have taken the opportunity to expand on several of our favorite examples because the machines now have the speed and power to explore them in greater detail.

Other improvements to the third edition include:

1. Throughout the text, we have attempted to eliminate redundant examples and added several interesting ones. The following changes are especially worth noting.
   (a) In Chapter 2, First-Order Ordinary Differential Equations, we present the integrating factor approach, variation of parameters, and method of undetermined coefficients when solving first-order linear equations.
   (b) In Chapter 3, we discuss the Logistic difference equation and give some surprisingly simple ways to generate the classic "Pitchfork diagram" with Mathematica.
   (c) Chapter 4, Higher-Order Equations, has been completely reorganized; a new section on nonlinear equations has been added.
   (d) Chapter 5, Applications of Higher-Order Equations, has also been completely reorganized. The catenary is now included in the Other Applications section.
   (e) Chapter 6, Systems of Ordinary Differential Equations, includes several new examples. See especially Example 6.2.5.
   (f) Chapter 7, Applications of Systems, includes several new examples. See especially Examples 7.3.3, 7.3.4, and 7.3.6.
   (g) We have included references that we find particularly interesting in the **Bibliography**, even if they are not specific Mathematica-related texts. A comprehensive list of Mathematica-related publications can be found at the Wolfram website.

`http://store.wolfram.com/catalog/books/.`

Finally, we must express our appreciation to those who assisted in this project. We would like to express appreciation to our editors, Tom Singer and Barbara Holland, and our production editor, Brandy Palacios, at Academic Press for providing a pleasant environment in which to work. In addition, Wolfram Research, especially Misty Mosely, have been most helpful in providing us up-to-date information about Mathematica. Finally, we thank those close to us, especially Imogene Abell, Lori Braselton, Ada Braselton, and Mattie Braselton for enduring with us the pressures of meeting a deadline and for graciously accepting our demanding work schedules. We certainly could not have completed this task without their care and understanding.

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August, 2003

Introduction to Differential Equations

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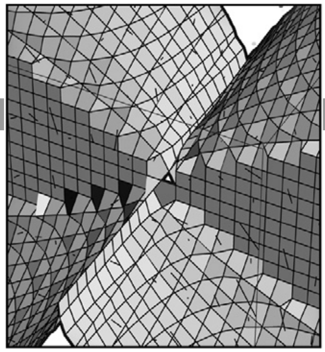

The purpose of *Differential Equations with Mathematica*, Third Edition, is twofold. First, we introduce and discuss the topics covered in typical undergraduate and beginning graduate courses in ordinary and partial differential equations including topics such as Laplace transforms, Fourier series, eigenvalue problems, and boundary-value problems. Second, we illustrate how Mathematica is used to enhance the study of differential equations not only by eliminating the computational difficulties, but also by overcoming the visual limitations associated with the explicit solutions to differential equations, which are often quite complicated. In each chapter, we first briefly present the material in a manner similar to most differential equations texts and then illustrate how Mathematica can be used to solve some typical problems. For example, in Chapter 2, we introduce the topic of first-order equations. First, we show how to solve certain types of problems by hand and then show how Mathematica can be used to assist in the same solution procedures. Finally, we illustrate how Mathematica commands like `DSolve` and `NDSolve` can be used to solve some frequently encountered equations exactly and/or numerically. In Chapter 3 we discuss some applications of first-order equations. Since we are experienced and understand the methods of solution covered in Chapter 2, we make use of `DSolve` and similar commands to obtain solutions. In doing so, we are able to emphasize the applications themselves as opposed to becoming bogged down in calculations.

The advantages of using Mathematica in the study of differential equations are numerous, but perhaps the most useful is that of being able to produce the graphics associated with solutions of differential equations. This is particularly beneficial in the discussion of applications because many physical situations are modeled with differential equations. For example, we will see that the motion of a pendulum can be modeled by a differential equation. When we solve the problem of the motion of a pendulum, we use technology to actually watch the pendulum move. The same is true for the motion of a mass attached to the end of a spring as well as many other problems. In having this ability, the study of differential equations becomes much more meaningful as well as interesting.

Numerous references like Abell and Braselton's *Mathematica By Example* [1] are also available to beginning users of Mathematica.

If you are a beginning Mathematica user and, especially, new to Version 5.0, the **Appendix** contains an introduction to Mathematica, including discussions about entering and evaluating commands, loading packages, and taking advantage of Mathematica's extensive help facility.

Although Chapter 1 is short in length, Chapter 1 introduces examples that will be investigated in subsequent chapters. Also, the vocabulary introduced in Chapter 1 will be used throughout the text. Consequently, even though, to a large extent, it may be read quickly, subsequent chapters will take advantage of the terminology and techniques discussed here.

1.1 Definitions and Concepts

We begin our study of differential equations by explaining what a differential equation is.

Definition 1 (Differential Equation). *A* ***differential equation*** *is an equation that contains the derivative or differentials of one or more dependent variables with respect to one or more independent variables. If the equation contains only ordinary derivatives (of one or more dependent variables) with respect to a single independent variable, the equation is called an* ***ordinary differential equation****.*

EXAMPLE 1.1.1: Thus, $dy/dx = x^2/\left(y^2 \cos y\right)$ and $dy/dx + du/dx = u + x^2 y$ are examples of *ordinary differential equations*.

The equation $(y-1)dx + x\cos y\, dy = 1$ is an *ordinary differential equation* written in *differential form*.

Using *prime notation*, a *solution* of the *ordinary differential equation* $xy'' + xy' + \left(x^2 - n^2\right)y = 0$, which is called **Bessel's equation**, is a function $y = y(x)$ with the property that $x\, d^2y/dx^2 + x\, dy/dx + \left(x^2 - n^2\right)y$ is identically the 0 function.

On the other hand,

$$\begin{cases} \dfrac{dx}{dt} = (a - by)x \\ \dfrac{dy}{dt} = (-m + nx)y \end{cases} \tag{1.1}$$

where a, b, m, and n are positive constants, is a *system* of two ordinary differential equations, called the **predator–prey equations**. A *solution* consists of two functions $x = x(t)$ and $y = y(t)$ that satisfy **both** equations. Predator–prey models can exhibit *very* interesting behavior as we will see when we study systems in more detail.

See texts like Giordano, Weir, and Fox's *A First Course in Mathematical Modeling* [12] and similar texts for detailed descriptions of predator–prey models.

Note that a system of differential equations can consist of more than two equations. For example, the basic equations that describe the competition between two organisms, with population densities x_1 and x_2, respectively, in a chemostat are

See Smith and Waltman's *The Theory of the Chemostat* [24] for a detailed discussion of chemostat models.

$$\begin{cases} S' = 1 - S - \dfrac{m_1 S}{a_1 + S} x_1 - \dfrac{m_2 S}{a_2 + S} x_2 \\ x_1{}' = x_1 \left(\dfrac{m_1 S}{a_1 + S} - 1 \right) \\ x_2{}' = x_2 \left(\dfrac{m_2 S}{a_2 + S} - 1 \right) \end{cases} \tag{1.2}$$

where $'$ denotes differentiation with respect to t; $S = S(t)$, $x_1 = x_1(t)$, and $x_2 = x_2(t)$. For equations (1.2), we remark that S denotes the concentration of the nutrient available to the competitors with population densities x_1 and x_2. We investigate chemostat models in more detail in Chapter 9.

If the equation contains partial derivatives of one or more dependent variables, then the equation is called a **partial differential equation**.

EXAMPLE 1.1.2: Because the equations involve partial derivatives of an unknown function, equations like $u\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}$ and $uu_x + u = u_{yy}$ are partial differential equations. For **Laplace's equation**, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ a *solution* would be a function $u = u(x, y)$ such that $u_{xx} + u_{yy}$ is identically the 0 function. A *solution* $u = u(x, t)$ of the **wave equation** is a function satisfying $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$.